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Two generalizations of some fixed point theorems

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ABSTRACT

The aim of this paper is to generalize two classical fixed point theorems given by Bogin [J. Bogin, A generalization of a fixed point theorem of Goebel, Kirk and Shimi, Canad. Math. Bull. 19 (1976) 7–12] and Greguš [M. Greguš, A fixed point theorem in Banach spaces, Boll. Un. Math. Ital. A (5) 17 (1980) 193–198]. We also complement and extend some very recent results proved by Suzuki [T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc. 136 (2008) 1861–1869].

1. Introduction

Let (X, d) be a metric space, T a self-mapping on X and k a nonnegative real number such that the inequality

$$d(Tx, Ty) \leq kd(x, y)$$

holds for any $x, y \in X$. If $k < 1$ then T is said to be a contractive mapping, if $k = 1$, then T is said to be a nonexpansive mapping. The well-known Banach theorem states that if X is complete, then every contractive mapping has a unique fixed point and that point can be obtained as a limit of repeated iteration of the mapping at any point of X. However, a nonexpansive mapping need not have fixed points. Yet these mappings have a fixed point when X has a convex structure. There exists a huge literature about contractive and nonexpansive type mappings, where the contractive and nonexpansive conditions are replaced with more general conditions (see, for instance [1–13]).

Bogin [1] proved the following result.

Theorem 1.1. *Let (X, d) be a nonempty complete metric space and $T : X \to X$ a mapping satisfying*

$$d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)], \tag{1}$$

where $a \geq 0, b > 0, c > 0$ and $a + 2b + 2c = 1$. Then T has a unique fixed point.

This result was generalized by Ćirić [3] and Li [14]. Greguš [9] considered a class of self-mapping T on X which satisfy (1) and (2) with $c = 0$. He proved the following theorem.

Theorem 1.2. *Let C be a nonempty closed convex subset of a Banach space B and $T : C \to C$ a mapping that satisfies*

$$\|Tx - Ty\| \leq a \|x - y\| + b (\|x - Tx\| + \|y - Ty\|) \tag{2}$$

for all $x, y \in C$, where $a > 0, b > 0$ and $a + 2b = 1$. Then T has a unique fixed point.

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Greguš's result has inspired many authors in further investigations: Abdeljawad and Karapinar [15], Ćirić [16–18], Delbosco et al. [19], Diviccaro et al. [7], Fisher [8], Fisher and Sessa [20], Jungck [10], Li [14], Murthy et al. [21], Mukherjee and Verma [22], Raswan and Ahmed [23]. Ćirić [24] has constructed an example to show that if the mapping T satisfies (1) with $b = 0$ and if a and c are such that (2) holds, then T need not have a fixed point.

The following remarkable generalization of the classical Banach contraction theorem, due to Suzuki [25], has lead to some important contributions in metric fixed point theory (see, for instance, [26–33]).

Theorem 1.3. *Let (X, d) be a complete metric space and $S : X \to X$. Define a nonincreasing function θ from $[0, 1)$ onto $\left(\frac{1}{2}, 1\right]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le \left(\sqrt{5}-1\right)/2, \\ (1-r)/r^2 & \text{if } \left(\sqrt{5}-1\right)/2 \le r \le 1/\sqrt{2}, \\ 1/(1+r) & \text{if } 1/\sqrt{2} \le r < 1. \end{cases}$$

Assume that there exists $r \in [0, 1)$ such that $\theta(r)d(x, Sx) \le d(x, y)$ implies $d(Sx, Sy) \le rd(x, y)$, for all $x, y \in X$. Then S has a unique fixed point.

Also, Kikkawa and Suzuki [34] proved Kannan and Meir–Keeler's versions of Theorem 1.3. Moreover, Suzuki studied a class of operators satisfying the following condition.

Definition 1.4. Let T be a mapping on a subset C of a Banach space E. Then T is said to satisfy "condition" (C) if for all $x, y \in C$

$$\text{(C)} \quad 1/2\,\|x - Tx\| \le \|x - y\| \text{ implies } \|Tx - Ty\| \le \|x - y\|.$$

The condition (C) is weaker than nonexpansiveness.

Inspired by these results we give in this paper two generalizations of Theorems 1.1 and 1.2.

2. Main results

The first result of this paper is the following generalization of Bogin's theorem.

Theorem 2.1. *Let (X, d) be a nonempty complete metric space and $T : X \to X$ be a mapping satisfying*

$$1/2d(x, Tx) \le d(x, y)$$

implies

$$d(Tx, Ty) \le ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)], \tag{3}$$

where $a \ge 0$, $b > 0$, $c > 0$ and $a + 2b + 2c = 1$. Then T has a unique fixed point.

Proof. Let $x \in X$ be arbitrary. Since $1/2d(x, Tx) \le d(x, Tx)$ we have that

$$d(Tx, T^2x) \le ad(x, Tx) + b[d(x, Tx) + d(Tx, T^2x)] + c[d(x, T^2x) + d(Tx, Tx)].$$

Hence

$$d(Tx, T^2x) \le (a + b)d(x, Tx) + bd(Tx, T^2x) + c[d(x, Tx) + d(Tx, T^2x)].$$

Therefore we obtain

$$d(Tx, T^2x) \le \frac{a + b + c}{1 - b - c}d(x, Tx) = d(x, Tx).$$

This implies that the sequence $\{d_n\}_{n=0}^{\infty}$ is a decreasing one, where $d_n := d(T^nx, T^{n+1}x)$ and $T^0x := x$.

Next we will show that there exists a nonnegative number $m < 2$ such that $d(Tx, T^3x) \le md_0$. First, we suppose that $d(x, T^2x) \ge d(x, Tx)$. Then $1/2d(x, Tx) \le d(x, T^2x)$ and we have

$$d(Tx, T^3x) \le ad(x, T^2x) + bd(x, Tx) + bd(T^2x, T^3x) + cd(x, T^3x) + cd(Tx, T^2x).$$

Thus

$$d(Tx, T^3x) \le a(d_0 + d_1) + bd_0 + bd_2 + c[d_0 + d(Tx, T^3x)] + cd_1$$

and so,

$$d(Tx, T^3x) \le \frac{2a + 2b + 2c}{1 - c}d_0 = \frac{1 + a}{1 - c}d_0.$$

Setting $m_1 = (1 + a)/(1 - c)$, we have $m_1 < 2$ and $d(Tx, T^3x) \le m_1 d_0$. Now, we suppose that $d(x, T^2x) < d(x, Tx)$. Since

$$d(Tx, T^2x) \le ad(x, Tx) + b[d(x, Tx) + d(Tx, T^2x)] + c[d(x, T^2x) + d(Tx, Tx)],$$

we get

$$d_1 < (a + b)d_0 + bd_1 + cd_0.$$

Hence

$$d_1 < \frac{a + b + c}{1 - b} d_0$$

and then

$$d(Tx, T^3x) \le d(Tx, T^2x) + d(T^2x, T^3x) \le 2d(Tx, T^2x) \le \frac{2a + 2b + 2c}{1 - b} d_0 = \frac{1 + a}{1 - b} d_0.$$

Setting $m_2 = (1 + a)/(1 - b)$, we have $m_2 < 2$ and $d(Tx, T^3x) \le m_2 d_0$. Taking $m = \max\{m_1, m_2\}$, we get $0 < m < 2$ and $d(Tx, T^3x) \le md(x, Tx)$, for all $x \in X$.

Since $1/2d(Tx, T^2x) \le d(Tx, T^2x)$ we have

$$d(T^2x, T^3x) \le ad(Tx, T^2x) + b[d(Tx, T^2x) + d(T^2x, T^3x)] + cd(Tx, T^3x).$$

Thus

$$d_2 \le (a + 2b)d_0 + mcd_0 = (a + 2b + mc)d_0.$$

Setting $k = a + 2b + mc$, we have $k < 1$ and $d_2 \le kd_0$ for all $x \in X$.

Let $x_0 \in X$ and $u_n = T^n x_0$. Then $d_{n+2} \le kd_n$ for all $n \ge 0$, where $d_n = d(u_n, u_{n+1})$. Therefore, for any even integer $n \ge 0$ we have by induction $d_n \le k^{n/2} d_0 \le k^{(n-1)/2} d_0$ and for every odd integer $n \ge 1$ we have also by induction $d_n \le k^{(n-1)/2} d_1 \le k^{(n-1)/2} d_0$. Hence, for all $n \ge 0$ we get $d_n \le k^{(n-1)/2} d_0$. Since $k \in (0, 1)$ we obtain that $\{u_n\}$ is a Cauchy sequence and by completeness of X we have that there exists $z \in X$ such that the sequence $\{u_n\}$ converges to z as $n \to \infty$.

Next, we will show that z is a fixed point of T. Assuming that there exists n such that $d(z, u_n) < 1/2d(u_n, u_{n+1})$ and $d(z, u_{n+1}) < 1/2d(u_{n+1}, u_{n+2})$ we obtain

$$d_n = d(u_n, u_{n+1}) \le d(z, u_n) + d(z, u_{n+1}) < 1/2(d_n + d_{n+1}) \le d_n.$$

This is a contradiction, so for all $n \ge 0$ we have either $d(z, u_n) \ge 1/2d(u_n, u_{n+1})$ or $d(z, u_{n+1}) \ge 1/2d(u_{n+1}, u_{n+2})$. Thus, there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $d(u_{n_j}, z) \ge 1/2d(u_{n_j+1}, u_{n_j})$ for every integer $j \ge 0$. Then, we have

$$d(Tz, u_{n_j+1}) \le ad(z, u_{n_j}) + bd(z, Tz) + bd(u_{n_j}, u_{n_j+1}) + cd(z, u_{n_j+1}) + cd(Tz, u_{n_j}).$$

Taking $j \to \infty$ we get $d(Tz, z) \le (b + c)d(Tz, z)$. This implies $d(Tz, z) = 0$ and so, $Tz = z$.

If y is another fixed point T then $d(y, z) \ge 1/2d(z, Tz) = 0$ and then

$$d(y, z) = d(Ty, Tz) \le ad(y, z) + b[d(y, y) + d(z, z)] + c[d(y, z) + d(y, z)].$$

Hence $d(y, z) \le (a + 2c)d(y, z)$. This implies $d(y, z) = 0$, which is a contradiction. So, T has a unique fixed point. □

Example 2.2. Let $X = [-1, 1]$ with the usual metric and let $T : X \to X$ be given as

$$Tx = \begin{cases} -x & \text{if } x \in [0, 3/4) \cup (3/4, 1] = U, \\ x/2 & \text{if } x \in [-1, 0) = V, \\ 0 & \text{if } x = 3/4. \end{cases}$$

We will prove that:

1. T has a unique fixed point.
2. T satisfies condition (3) with $a = 1/2$, $b = c = 1/8$, i.e.,

   $$d(x, Tx)/2 \le d(x, y) \Rightarrow d(Tx, Ty) \le M(x, y),$$

   where $M(x, y) = \frac{1}{2}d(x, y) + \frac{1}{8}[d(x, Tx) + d(y, Ty) + d(x, Ty) + d(y, Tx)]$.
3. T does not satisfy Suzuki's condition from Theorem 1.3.
4. T does not satisfy Bogin's condition (1) with $a = 1/2$, $b = c = 1/8$.

Proof. 1 is obvious. 2.

(i) For $x, y \in U$ it is $M(x,y) = \frac{1}{2}|y-x| + \frac{1}{8}(2x+2y+x+y+x+y) = \frac{1}{2}|y-x| + \frac{1}{2}(x+y) \geq |y-x| = d(Tx,Ty)$ and (3) holds.

(ii) If $x, y \in V$, then $M(x,y) \geq \frac{1}{2}d(x,y) = \frac{1}{2}|y-x| = d(Tx,Ty)$, so (3) holds.

(iii) If $x \in U, y \in V$, then $M(x,y) = \frac{1}{2}(x-y) + \frac{1}{8}\left(2x - \frac{1}{2}y + x - \frac{1}{2}y + |y+x|\right) = \frac{7}{8}x - \frac{5}{8}y + \frac{1}{8}|y+x| \geq x - \frac{1}{2}y$, $d(Tx,Ty) = \left|x + \frac{1}{2}y\right|$. Since $x \geq 0$, $y < 0$ we have $x - \frac{1}{2}y \geq x + \frac{1}{2}y$ and $x - \frac{1}{2}y \geq -x - \frac{1}{2}y$, so (3) holds.

(iv) If $x \in V, y \in U$, then $M(x,y) \geq d(Tx,Ty)$ like in (iii).

(v) For $x \in U$, $y = \frac{3}{4}$, it is $M(x,y) = \frac{1}{2}\left|x - \frac{3}{4}\right| + \frac{1}{8}\left(2x + \frac{3}{4} + x + \frac{3}{4} + x\right) = \frac{1}{2}\left|x - \frac{3}{4}\right| + \frac{1}{2}x + \frac{3}{16}$ and $d(Tx,Ty) = x$. Since $d(x,Tx)/2 \leq d(x,y)$ we have $x \leq \left|x - \frac{3}{4}\right|$, so $x \leq \frac{3}{8}$. Therefore $M(x,y) = \frac{3}{8} + \frac{3}{16} = \frac{9}{16} \geq x$, and (3) holds.

(vi) For $x \in V$, $y = \frac{3}{4}$, it is $M(x,y) = \frac{1}{2}\left(\frac{3}{4} - x\right) + \frac{1}{8}\left(-\frac{x}{2} + \frac{3}{4} - x + \frac{3}{4} - \frac{x}{2}\right) = -\frac{3x}{4} + \frac{9}{16} \geq \frac{1}{2}$, and $d(Tx,Ty) = -\frac{x}{2} \leq \frac{1}{2}$. Then (3) holds.

(vii) If $x = \frac{3}{4}$, $y \in U$, then $M(x,y) = \frac{1}{2}\left|\frac{3}{4} - y\right| + \frac{1}{8}\left(\frac{3}{4} + 2y + y + \frac{3}{4} + y\right) = \frac{1}{2}\left|\frac{3}{4} - y\right| + \frac{y}{2} + \frac{3}{16}$ and $d(Tx,Ty) = y$. By $d(x,Tx)/2 \leq d(x,y)$ we have $\frac{3}{8} \leq \left|\frac{3}{4} - y\right|$, so $y \leq \frac{3}{8}$. Therefore $M(x,y) = \frac{3}{8} + \frac{3}{16} = \frac{9}{16} \geq y$. Hence (3) holds.

(viii) If $x = \frac{3}{4}$, $y \in V$, then $M(x,y) = \frac{1}{2}\left(\frac{3}{4} - y\right) + \frac{1}{8}\left(\frac{3}{4} - \frac{y}{2} - y + \frac{3}{4} - \frac{y}{2}\right) = \frac{9}{16} - \frac{3y}{4} \geq \frac{1}{2}$ and $d(Tx,Ty) = -\frac{y}{2} \leq \frac{1}{2}$ $\left(\frac{y}{2} \in \left[-\frac{1}{2}, 0\right)\right)$. Hence (3) holds.

(ix) If $x = y$ then (3) is obvious.

3. If $x = 0$, $y = 1$, then $\theta(r)d(x,Tx) = 0 < 1 = d(x,y)$ and $d(Tx,Ty) = 1$, so condition from Theorem 1.3. does not hold.

4. If $x = \frac{3}{4}$, $y = 1$ we have $d(Tx,Ty) = 1$ and $M(x,y) = \frac{1}{8} + \frac{1}{8}\left(\frac{3}{4} + 2 + 1 + 1 + \frac{3}{4}\right) = \frac{13}{16}$, so $d(Tx,Ty) > M(x,y)$. Therefore Bogin's condition (2) does not hold. □

For the next theorem we need recall an important concept.

Definition 2.3 (*Takahashi [35]*)**.** Let (X,d) be a metric space. A mapping $W : X \times X \times [0,1] \to X$ is said to be a *convex structure* on X if for each $(x,y,\lambda) \in X \times X \times [0,1]$ and $u \in X$

$$d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y).$$

The metric space X together with W is called a *convex metric space*.

It is obvious that in a convex metric space we have $d(x, W(x,y,\lambda)) = (1-\lambda)d(x,y)$ and $d(y, W(x,y,\lambda)) = \lambda d(x,y)$. Now we can prove a generalization of Greguš' s theorem.

Theorem 2.4. *Let X be a complete convex metric space and let $T : X \to X$ be a selfmapping. Assume that $1/2d(x,Tx) \leq d(x,y)$ implies*

$$d(Tx,Ty) \leq ad(x,y) + bd(x,Tx) + bd(y,Ty), \tag{4}$$

where $a > 0$, $b > 0$ and $a + 2b = 1$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$, $u_n = T^n x_0$ and $d_n = d(u_n, u_{n+1})$. Since $1/2d(T^n x_0, T^{n+1}x_0) \leq d(T^n x_0, T^{n+1}x_0)$, we have from the assumption that

$$d_{n+1} \leq ad_n + bd_n + bd_{n+1}.$$

Thus

$$d_{n+1} \leq \frac{a+b}{1-b}d_n = d_n,$$

for all $n \geq 0$. So, $\{d_n\}$ is a decreasing sequence.

Let $t = W(Tx_0, T^2x_0, 1/2)$ and $z = W(T^2x_0, T^3x_0, 1/2)$. Suppose that $d(Tx_0, z) < 1/2d(Tx_0, T^2x_0)$. Since

$$d(T^2x_0, z) = 1/2d(T^2x_0, T^3x_0) \leq 1/2d(Tx_0, T^2x_0)$$

it follows that

$$d(Tx_0, T^2x_0) \leq d(Tx_0, z) + d(T^2x_0, z) < d(Tx_0, T^2x_0),$$

which is a contradiction. Thus we have $d(Tx_0, z) \geq 1/2d(Tx_0, T^2x_0)$. From the assumption we obtain

$$d(T^2x_0, Tz) \leq ad(Tx_0, z) + bd(Tx_0, T^2x_0) + bd(z, Tz).$$

Since $1/2d(T^2x_0, T^3x_0) = d(T^2x_0, z) \le d(T^2x_0, z)$, we also obtain

$$d(T^3x_0, Tz) \le ad(T^2x_0, z) + bd(T^2x_0, T^3x_0) + bd(z, Tz).$$

Therefore

$$\begin{aligned} d(z, Tz) &\le 1/2d(Tz, T^2x_0) + 1/2d(Tz, T^3x_0) \\ &\le 1/2[ad(Tx_0, z) + bd(Tx_0, T^2x_0) + bd(z, Tz)] + 1/2[ad(T^2x_0, z) + bd(T^2x_0, T^3x_0) + bd(z, Tz)]. \end{aligned}$$

Thus we get

$$(2 - 2b)d(z, Tz) \le ad(z, Tx_0) + ad(z, T^2x_0) + 2bd(x_0, Tx_0),$$

and so,

$$\begin{aligned} d(z, Tz) &\le \frac{a}{1+a}d(z, Tx_0) + \frac{a}{2(1+a)}d(T^2x_0, T^3x_0) + \frac{2b}{1+a}d_0 \\ &\le \frac{a}{1+a}d(z, Tx_0) + \frac{a}{2(1+a)}d_0 + \frac{2b}{1+a}d_0 \\ &= \frac{a}{1+a}d(z, Tx_0) + \frac{2-a}{2(1+a)}d_0. \end{aligned}$$

Next we consider two cases.

Case 1. Suppose that $1/2d(x_0, Tx_0) \le d(x_0, T^2x_0)$. By the assumption we get

$$\begin{aligned} d(T^3x_0, Tx_0) &\le ad(x_0, T^2x_0) + bd(x_0, Tx_0) + bd(T^2x_0, T^3x_0) \\ &\le a[d(x_0, Tx_0) + d(Tx_0, T^2x_0)] + 2bd_0 \\ &\le (2a + 2b)d_0 = (1 + a)d_0. \end{aligned}$$

Since $d(z, Tx_0) \le 1/2d(Tx_0, T^2x_0) + 1/2d(Tx_0, T^3x_0)$ we obtain $d(z, Tx_0) \le 1/2(2 + a)d_0$. Therefore

$$d(z, Tz) \le \frac{a}{1+a} \cdot \frac{2+a}{2}d_0 + \frac{2-a}{2(1+a)}d_0 = \frac{a^2+a+2}{2(1+a)}d_0.$$

Since $a^2 + a + 2 < 2 + 2a$, taking

$$k_1 = \frac{a^2+a+2}{2(1+a)}$$

we have $k_1 < 1$ and $d(z, Tz) \le k_1d_0$.

Case 2. Suppose that $1/2d(x_0, Tx_0) > d(x_0, T^2x_0)$. We will show that there exists $k_2 < 1$ such that $d(t, Tt) \le k_2d_0$. Assume that $d(x_0, t) < 1/2d(x_0, Tx_0)$. If $d(Tx_0, t) \le 1/2d(x_0, Tx_0)$ then we have $d(x_0, Tx_0) \le d(x_0, t) + d(Tx_0, t) < d(x_0, Tx_0)$, which is a contradiction. So, we get $d(Tx_0, t) > 1/2d(x_0, Tx_0)$. Since $d(Tx_0, t) = 1/2d(Tx_0, T^2x_0)$, we obtain that $d(Tx_0, T^2x_0) > d(x_0, Tx_0)$, which is also a contradiction. Therefore, we must have $1/2d(x_0, Tx_0) \le d(x_0, t)$. By the assumption we get

$$d(Tx_0, Tt) \le ad(x_0, t) + bd(t, Tt) + bd(x_0, Tx_0).$$

Since $1/2d(Tx_0, T^2x_0) = 1/2d(x_0, t) \le d(x_0, t)$ we also get

$$d(T^2x_0, Tt) \le ad(Tx_0, t) + bd(t, Tt) + bd(Tx_0, T^2x_0).$$

Hence we obtain

$$\begin{aligned} d(t, Tt) &\le 1/2d(Tx_0, Tt) + 1/2d(T^2x_0, Tt) \\ &\le 1/2[ad(x_0, t) + bd(t, Tt) + bd(x_0, Tx_0)] + 1/2[ad(Tx_0, t) + bd(t, Tt) + bd(Tx_0, T^2x_0)]. \end{aligned}$$

Thus

$$\begin{aligned} (1 - b)d(t, Tt) &\le a/2d(x_0, t) + a/2d(Tx_0, t) + bd(x_0, Tx_0) \\ &= a/2d(x_0, t) + a/4d(Tx_0, T^2x_0) + bd(x_0, Tx_0) \\ &\le a/2d(x_0, t) + (a/4 + b)d_0. \end{aligned}$$

Since $d(x_0, t) \le 1/2d(x_0, Tx_0) + 1/2d(x_0, T^2x_0) < 1/2d_0 + 1/4d_0 = 3/4d_0$, we obtain

$$(1 - b)d(t, Tt) \le (3a/8 + a/4 + b)d_0,$$

which implies

$$d(t, Tt) \le \frac{5/8a + b}{1 - b} d_0.$$

But $5/8a + b < 1 - b$, so taking

$$k_2 = \frac{5/8a + b}{1 - b}$$

we have $k_2 < 1$ and $d(t, Tt) \le k_2 d_0$.

Hence in all cases we proved that there exists $k < 1$ and $y \in X$ such that $d(y, Ty) < kd(x_0, Tx_0)$. Therefore $\inf\{d(x, Tx) : x \in X\} = 0$.

Next, we will prove that the infimum is attained. Take the following system of sets:

$K_n := \{x \in X : d(x, Tx) \le r/(2n)\}$, TK_n and $\overline{TK_n}$, where $\overline{TK_n}$ is the closure of TK_n, $r = (1 - a)/(1 + b),\ n \ge 1$.

For any $x, y \in K_n$ such that $1/2d(x, Tx) \le d(x, y)$ we have

$$\begin{aligned} d(x, y) &\le d(x, Tx) + d(Tx, Ty) + d(y, Ty) \\ &\le \frac{r}{2n} + ad(x, y) + bd(x, Tx) + bd(y, Ty) + \frac{r}{2n} \\ &\le ad(x, y) + \frac{(1 + b)r}{n}. \end{aligned}$$

Thus

$$d(x, y) \le \frac{1 + b}{1 - a} \cdot \frac{r}{n} = \frac{1}{n}.$$

If $1/2d(x, Tx) > d(x, y)$ we have

$$d(x, y) < \frac{1}{2} \cdot \frac{r}{2n} < \frac{1}{n}.$$

Hence $d(x, y) \le 1/n$ for all $x, y \in K_n$, i.e. $\text{diam}(K_n) \le 1/n$. Since $d(Tx, T^2x) \le d(x, Tx)$ we have $T(K_n) \subseteq K_n$ and then, $\text{diam}(\overline{T(K_n)}) = \text{diam}(T(K_n)) \le 1/n$.

Supposing $y \in \overline{T(K_n)}$, then for any $\epsilon > 0$ there exists $y' \in K_n$ such that $d(y, Ty') < \epsilon$. If $d(y, y') < 1/2d(y', Ty')$ and $d(y, Ty') < 1/2d(Ty', T^2y')$, then

$$d(y', Ty') < d(y, y') + d(y, Ty') < 1/2[d(y', Ty') + d(Ty', T^2y')] \le d(y', Ty'),$$

which is a contradiction. Hence, either $d(y, y') \ge 1/2d(y', Ty')$ or $d(y, Ty') \ge 1/2d(Ty', T^2y')$. In the first case we have by the assumption

$$d(Ty, Ty') \le ad(y, y') + bd(y, Ty) + bd(y', Ty').$$

Thus

$$\begin{aligned} d(y, Ty) &\le d(y, Ty') + d(Ty, Ty') \\ &\le \epsilon + ad(y, y') + bd(y, Ty) + \frac{br}{2n} \\ &\le \epsilon + a[d(y, Ty') + d(Ty', y')] + bd(y, Ty) + \frac{br}{2n}. \end{aligned}$$

Hence

$$d(y, Ty) \le \frac{(a + 1)\epsilon}{1 - b} + \frac{a + b}{1 - b} \cdot \frac{r}{2n} < \frac{a + 3}{1 - b}\epsilon + \frac{r}{2n}.$$

In the second case, from the assumption we obtain

$$d(Ty, T^2y') \le ad(y, Ty') + bd(y, Ty) + bd(Ty', T^2y').$$

But $d(Ty', T^2y') \le 2d(y, Ty') = 2\epsilon$ and then

$$\begin{aligned} d(y, Ty) &\le d(y, T^2y') + d(T^2y', Ty) \le d(y, Ty') + d(Ty', T^2y') + d(T^2y', Ty) \\ &\le \epsilon + 2\epsilon + ad(y, Ty') + bd(y, Ty) + \frac{br}{2n} \\ &\le \epsilon + 2\epsilon + a\epsilon + bd(y, Ty) + \frac{br}{2n} \\ &\le (a+3)\epsilon + bd(y, Ty) + \frac{br}{2n}. \end{aligned}$$

Hence

$$d(y, Ty) \le \frac{a+3}{1-b}\epsilon + \frac{b}{1-b} \cdot \frac{r}{2n} < \frac{a+3}{1-b}\epsilon + \frac{r}{2n}.$$

Therefore, in all cases we proved that

$$d(y, Ty) \le \frac{a+3}{1-b}\epsilon + \frac{r}{2n}.$$

Since $\epsilon > 0$ is arbitrary, we get that $d(y, Ty) \le r/(2n)$, i.e. $y \in K_n$. Hence we have $\overline{T(K_n)} \subseteq K_n$. Therefore, $\left\{\overline{T(K_n)}\right\}$ is a decreasing sequence of closed nonempty sets with $\operatorname{diam}(\overline{T(K_n)}) \to 0$ as $n \to \infty$. Thus they have a nonempty intersection. Since $\overline{T(K_n)} \subseteq K_n$, we obtain that there exists $u \in K_n$ for all n. This implies that u is a fixed point of T.

If v is another fixed point of T, since $1/2d(u, Tu) = 0 \le d(u, v)$ we have

$$d(u, v) = d(Tu, Tv) \le ad(u, v) + bd(u, u) + bd(v, v) = ad(u, v),$$

which is a contradiction. □

Example 2.5. Let $X = [-1, 1]$ with the usual metric and let $T : X \to X$ be given as

$$Tx = \begin{cases} -x & \text{if } x \in [0, 2/3) \cup (2/3, 1] = U, \\ x/2 & \text{if } x \in [-1, 0) = V, \\ -1/4 & \text{if } x = 2/3. \end{cases}$$

We will prove that:

1. T has a unique fixed point.
2. T satisfies condition (4) with $a = 2/3$, $b = 1/6$, i.e.,
$$d(x, Tx)/2 \le d(x, y) \Rightarrow d(Tx, Ty) \le m(x, y),$$
where $m(x, y) = \frac{2}{3}d(x, y) + \frac{1}{6}[d(x, Tx) + d(y, Ty)]$.
3. T does not satisfy Suzuki's condition from Theorem 1.3.
4. T does not satisfy Greguš's condition (2) with $a = 2/3$, $b = 1/6$.

Proof. 1 is obvious. 2.

(i) For $x, y \in U$ it is $m(x, y) = \frac{2}{3}|y - x| + \frac{2x}{6} + \frac{2y}{6} \ge \frac{2}{3}|y - x| + \frac{1}{3}|y - x| = |y - x| = d(Tx, Ty)$ and (4) holds.

(ii) If $x, y \in V$, then $m(x, y) = \frac{2}{3}|y - x| + \frac{1}{6}\left|\frac{x}{2}\right| + \frac{1}{6}\left|\frac{y}{2}\right| \ge \frac{1}{2}|y - x| = d(Tx, Ty)$, so (4) holds.

(iii) If $x \in U, y \in V$, then $m(x, y) = \frac{2}{3}|y - x| + \frac{2x}{6} + \frac{1}{6}\left|\frac{y}{2}\right| = \frac{2}{3}(x - y) + \frac{1}{3}x - \frac{1}{12}y = x - \frac{3}{4}y$, $d(Tx, Ty) = \left|x + \frac{1}{2}y\right|$. Since $x \ge 0, y < 0$ we have $x - \frac{3}{4}y \ge x + \frac{1}{2}y$ and $x - \frac{3}{4}y \ge -x - \frac{1}{2}y$, so (4) holds.

(iv) If $x \in V, y \in U$, then $m(x, y) \ge d(Tx, Ty)$ like in (iii).

(v) For $x \in U, y = \frac{2}{3}$, it is $m(x, y) = \frac{2}{3}\left|x - \frac{2}{3}\right| + \frac{2x}{6} + \frac{1}{6}\left(\frac{2}{3} + \frac{1}{4}\right)$ and $d(Tx, Ty) = \left|x - \frac{1}{4}\right|$. Since $d(x, Tx)/2 \le d(x, y)$ we have $x \le \left|x - \frac{2}{3}\right|$, so $x \le \frac{1}{3}$. Therefore $\left|x - \frac{1}{4}\right| \le \frac{1}{4}$ and $\left|x - \frac{2}{3}\right| \ge \frac{1}{3}$. Hence $m(x, y) \ge \frac{2}{9} + \frac{1}{6}\left(\frac{2}{3} + \frac{1}{4}\right) = \frac{3}{8} > d(Tx, Ty)$, and (4) holds.

(vi) For $x \in V, y = \frac{2}{3}$, it is $m(x, y) = \frac{2}{3}\left(\frac{2}{3} - x\right) + \frac{1}{6}\left|\frac{x}{2}\right| + \frac{1}{6}\left(\frac{2}{3} + \frac{1}{4}\right) \ge \frac{4}{9} > \frac{1}{4}$, and $d(Tx, Ty) = \left|\frac{x}{2} + \frac{1}{4}\right| \le \frac{1}{4}$ $\left(\frac{x}{2} \in \left[-\frac{1}{2}, 0\right)\right)$, so (4) holds.

(vii) If $x = \frac{2}{3}, y \in U$, then $m(x, y) = \frac{2}{3}\left|\frac{2}{3} - y\right| + \frac{1}{6}\left(\frac{2}{3} + \frac{1}{4}\right) + \frac{2y}{6}$ and $d(Tx, Ty) = \left|y - \frac{1}{4}\right|$. By $d(x, Tx)/2 \le d(x, y)$ we have $\frac{1}{2}\left(\frac{2}{3} + \frac{1}{4}\right) = \frac{11}{24} \le \left|y - \frac{2}{3}\right|$, so $y \le \frac{5}{24}$. Therefore $m(x, y) \ge \frac{2}{3} \times \frac{11}{24} > \frac{1}{4}$ and $d(Tx, Ty) \le \frac{1}{4}$. Hence (4) holds.

(viii) If $x = \frac{2}{3}, y \in V$, then $m(x, y) \ge \frac{2}{3}d(x, y) = \frac{2}{3}\left(\frac{2}{3} - y\right) \ge \frac{4}{9} > \frac{1}{4}$ and $d(Tx, Ty) = \left|-\frac{1}{4} - \frac{y}{2}\right| \le \frac{1}{4}$ $\left(\frac{y}{2} \in \left[-\frac{1}{2}, 0\right)\right)$. Hence (4) holds.

(ix) If $x = y$ then (4) is obvious.

3. If $x = 0, y = 1$, then $\theta(r)d(x, Tx) = 0 < 1 = d(x, y)$ and $d(Tx, Ty) = 1$, so condition from Theorem 1.3. does not hold.

4. If $x = \frac{2}{3}, y = 1$ we have $d(Tx, Ty) = \frac{3}{4}$ and $m(x, y) = \frac{2}{9} + \frac{2}{6} + \frac{1}{6}\left(\frac{2}{3} + \frac{1}{4}\right) = \frac{51}{72} < \frac{54}{72} = \frac{3}{4}$, so $d(Tx, Ty) > m(x, y)$. Therefore Greguš's condition (2) does not hold. □

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